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Utility Functions for Debreu's 'Excess Demands'

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In a series of papers by Sonnenschein (1973), Mantel (1973), Debreu (1974), McFadden, Mas-Collel, Mantel, Richter (1974), and Mantel (1976), it has been demonstrated that neoclassical micro-economic theory imposes almost no restriction on community excess demand functions other than Walras Law, if the economy contains no more commodities than consumers. Unfortunately, many of the ideas in these important proofs are hidden by the extremely complicated nature of the constructions.

Perhaps the most remarkable of these proofs is Debreu's. He decomposes an arbitrary continuous function $x(p)$ on $\dot{R}_{+}^{\ell}$ (the "candidate excess demand") satisfying Walras Law and homogeneity of degree zero into ℓ functions $\bar{x}^k(p)$, $k=1,\ldots,\ell$ and he constructs ℓ systems of convex, monotonic indifference surfaces in such a way that subject to the budget constraint $p^t x \leq 0$, $\bar{x}^k(p)$ lies on the highest indifference surface of the k^{th} system, $k=1,\ldots,\ell$, (so long as p is not too close to the boundary where some price is zero). In this paper I write down explicit concave and monotonic utility functions u^k, $k=1,\ldots,\ell$, constructed on the basis of a simple geometric intuition, such that maximizing the k^{th} utility function u^k subject to the budget constraint $p^t x \leq 0$ gives exactly the k^{th} individual excess demand $\bar{x}^k(p)$ in Debreu's decomposition, (away from the boundary). In order to guarantee concavity and monotonicity of the utility functions I impose the restriction that the original $x(p)$ be differentiable as well as continuous, but I hope thereby to bring the ideas lying behind Debreu's decomposition into sharper focus.

The plan of the paper is as follows. First the arbitrary $x(p)$ is decomposed, $x(p) = \sum_{k=1}^{\ell} \bar{x}^k(p)$ as in the Debreu paper. Then utility functions u^k are given that are maximized uniquely at $\bar{x}^k(p)$, given the budget constraint $x\varepsilon\{x\varepsilon R^{\ell}|p^t x \leq 0\}$, for all $p\varepsilon \mathring{R}^{\ell}_+$ and $k=1,\ldots\ell$. In order to assure monotonicity and concavity, however, we must restrict our attention in part 3 to a compact set of prices P, for instance $\{p\varepsilon R^{\ell}|p_i \geq \varepsilon,\ i=1,\ldots\ell, |p| = 1\}$ for any $\varepsilon > 0$. Then each individual excess demand $\bar{x}^k(p)$ will lie in some large convex and compact subset X^k of R^{ℓ}. By perturbing the utilities and taking advantage of the differentiability assumption, it is shown that each utility u^k can be taken to be monotonic and strictly concave on X^k and still give rise to the same $\bar{x}^k(p)$. Finally in part 4 of the argument u^k is extended to all of R^{ℓ}, preserving concavity, monotonicity, and the excess demands for all $p\varepsilon P$, but perhaps not for p outside of P. Observe that since X^k is bounded, we can choose $w^k \varepsilon R^{\ell}_+$, the initial endowment vectors for $k=1,\ldots,\ell$, large enough so that the net trade space $R^{\ell}_+ - w^k$ contains X^k for all $k=1,\ldots,\ell$.

Notation:

Let $\mathring{R}^{\ell}_+ = \{p\varepsilon R^{\ell}|p_i > 0,\ i=1,\ldots,\ell\}$. We denote by $T(p)$ or $[p]^{\perp}$ the set $\{x\varepsilon R^{\ell}|p^t x = 0\}$ for all $p\varepsilon \mathring{R}^{\ell}_+$, where $p^t x$ means p transpose. We let $p\perp x$ mean $p^t x = 0$. If $X \subset \mathbb{R}^{\ell}$, $X^{\perp} = \{y\varepsilon \mathbb{R}^{\ell} \mid y \perp x \text{ for all } x\varepsilon X\}$.

Let e^k be the k^{th} standard basis vector, $k=1,\ldots,\ell$ and let $\Pi_{T(p)}y$ be the projection of y perpendicularly onto $T(p)$, i.e., in the direction p. Then $\Pi_{T(p)}y = [I - \frac{pp^t}{|p|^2}]y$.

We write $x > y$ iff $x_i \geq y_i$, $i=1,\ldots,\ell$ and $x \neq y$ and $x >> y$ iff $x_i > y_i$, $i = 1, \ldots, \ell$. Let $x(p)$ denote a function $x: \overset{\circ}{R}{}^{\ell}_{+} \to R^{\ell}$. We call $x(p)$ a <u>candidate aggregate excess demand</u> function if it satisfies:

1.) Homogeneity (H): $x(p) = x(\lambda p)$ for all $\lambda > 0$ and all $p \varepsilon \overset{\circ}{R}{}^{\ell}_{+}$
2.) Walras Law (W): $p^t x(p) = 0$ for all $p \varepsilon \overset{\circ}{R}{}^{\ell}_{+}$
3.) Twice continuous differentiability (C^2): $x(p)$ is C^2 on $\overset{\circ}{R}{}^{\ell}_{+}$

We define a rational agent as an ordered pair (u,X) satisfying:

1.) $X = R^{\ell}_{+} - w$, for some $w \varepsilon R^{\ell}_{+}$, that is X is the set of net trades.
2.) u is a function $u : X \to R$ which is monotonic,
$x > y \Rightarrow u(x) > u(y)$ and concave:
$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$ for all x and y and
$0 \leq \lambda \leq 1$
3.) Given $p \varepsilon \overset{\circ}{R}{}^{\ell}_{+}$, the agent always acts to maximize $u(x)$ such that $x \varepsilon X$, $p^t x \leq 0$. If $x(p)$ is the unique solution to the agent's maximization problem for all $p \varepsilon P \subset \overset{\circ}{R}{}^{\ell}_{+}$, and if $x(p)$ is C^2, then we call $x(p)$ a <u>rational individual excess demand</u> on P.

Recall that u is monotonic if $Du(x) >> 0$ and strictly quasi-concave if $y^t D^2 u(x) y < 0$, for all y such that $y^t Du(x) = 0$, for all $x \varepsilon X$.

<u>Theorem</u>: Let $x(p)$ be a candidate excess demand function, $x: \overset{\circ}{R}{}^{\ell}_{+} \to R^{\ell}$. and let P be a compact set, $P \subset \overset{\circ}{R}{}^{\ell}_{+}$. Then there exist ℓ rational agents

$\{(u^k, X^k), k = 1,\ldots,\ell\}$ giving rise to ℓ rational individual excess demands $\bar{x}^k(p)$ such that $\sum_{k=1}^{\ell} \bar{x}^k(p) = x(p)$ for all $p \in P \subset \overset{\circ}{R}{}^{\ell}_{+}$.

<u>Proof</u>: Part 1: <u>Decomposition</u>:

By Walras Law, $p^t x(p) = 0$ for all $p \in \overset{\circ}{R}{}^{\ell}_{+}$, *i.e.*, $x(p) \in T(p)$. Now for all $p \in \overset{\circ}{R}{}^{\ell}_{+}$, we can find a scalar $\theta(p)$ such that $x(p) + \theta(p)\frac{p}{\|p\|} \gg 0$. If $x(p)$ is C^2 and homogeneous, then θ can be chosen so as well.

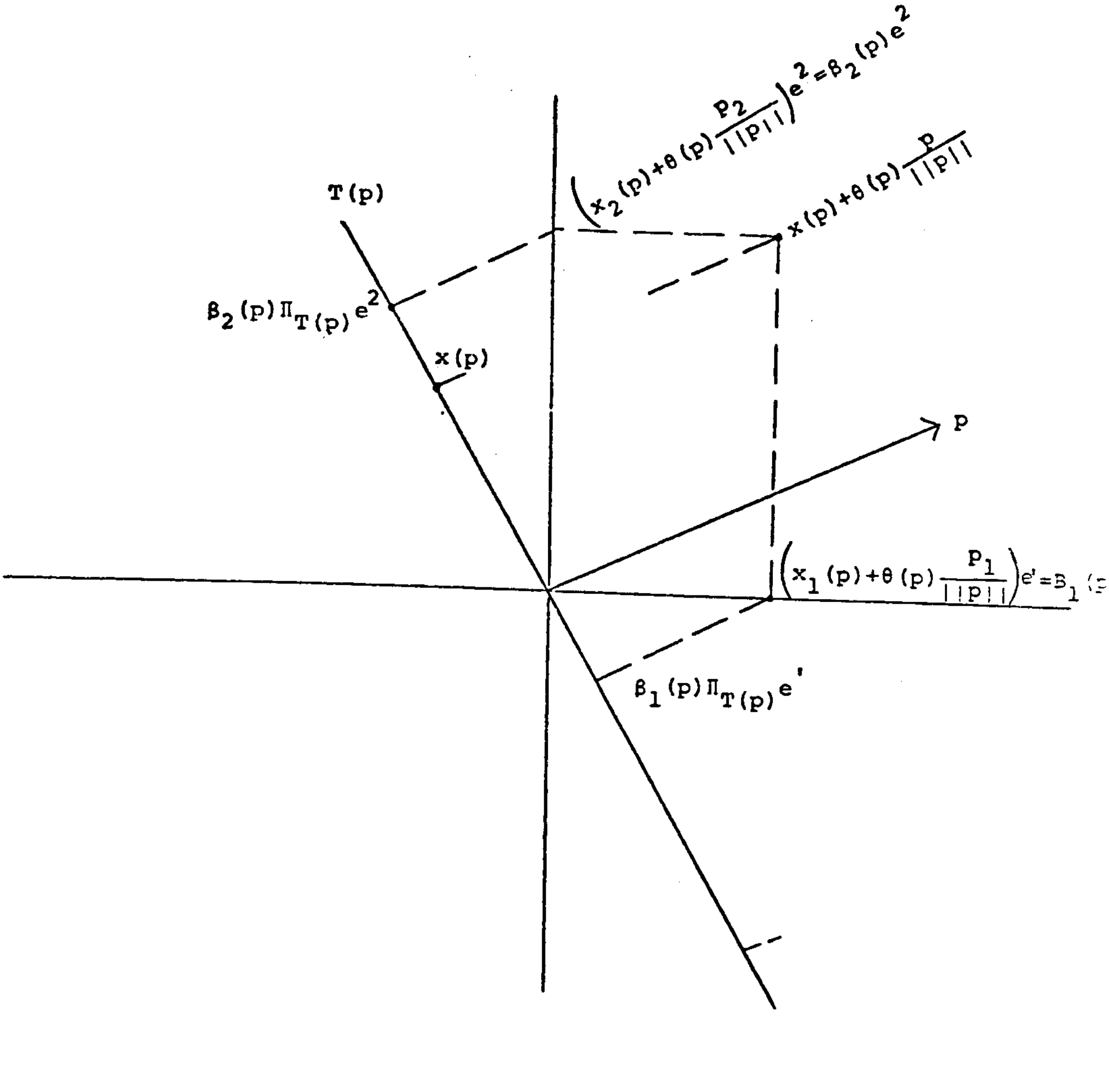
Let $\Pi_{T(p)}y$ denote the projection of y perpendicularly onto $T(p)$. If $\{e^1, e^2, \ldots, e^{\ell}\}$ is the standard basis for $\mathbb{R}^{\ell}$ and $x(p) = (x_1(p), \ldots, x_{\ell}(p))^t = \sum_{k=1}^{\ell} x_k(p)e^k$, then by noting that $\Pi_{T(p)}\theta(p)\frac{p}{\|p\|} = 0$, we can set

$$\begin{aligned} x(p) &= \Pi_{T(p)}x(p) = \Pi_{T(p)}\left[x(p) + \theta(p)\frac{p}{\|p\|}\right] \\ &= \Pi_{T(p)}\left[\sum_{k=1}^{\ell}\left(x_k(p) + \theta(p)\frac{p_k}{\|p\|}\right)e^k\right] \\ &= \sum_{k=1}^{\ell}\left(x_k(p) + \theta(p)\frac{p_k}{\|p\|}\right)\Pi_{T(p)}e^k \quad . \end{aligned}$$

Let $\beta_k(p) = x_k(p) + \theta(p)\frac{p_k}{\|p\|}$ and $\bar{x}^k(p) = \beta_k(p)\Pi_{T(p)}e^k$, $k = 1,\ldots,\ell$. We have just shown that $x(p)$ can be decomposed at all $p \in \overset{\circ}{\mathbb{R}}{}^{\ell}_{+}$ into ℓ functions $\bar{x}^k(p) = \beta_k(p)\Pi_{T(p)}e^k$, $k = 1,\ldots,\ell$ satisfying H, W, and C^2 and that $\beta_k(p) > 0$ for all $p \in \overset{\circ}{\mathbb{R}}{}^{\ell}_{+}$, and $k = 1,\ldots,\ell$.

$$x(p) = \sum_{k=1}^{\ell} \bar{x}^k(p) = \sum_{k=1}^{\ell} \beta_k(p)\Pi_{T(p)}e^k \quad .$$

Observe that $\Pi_{T(p)}e^k$ is a vector with a strictly positive k^{th} coordinate and $\ell-1$ strictly negative components, that is, $x_i^k(p) < 0$,



$\left(x_2(p)+\theta(p)\frac{p_2}{||p||}\right)e^2=\beta_2(p)e^2$
$x(p)+\theta(p)\frac{p}{||p||}$
T(p)
$\beta_2(p)\Pi_{T(p)}e^2$
x(p)
p
$\left(x_1(p)+\theta(p)\frac{p_1}{||p||}\right)e'=\beta_1(p)$
$\beta_1(p)\Pi_{T(p)}e'$

Figure 1

$i \neq k$ and $0 < x_k^k(p)$ for $k = 1, \ldots, \ell$. Accordingly, let

$\tilde{X}^k = \{x \in \mathbb{R}^\ell \mid x_i < 0,\ i \neq k,\ x_k > 0\}$.

Note that so far the only use of the continuity of $x(p)$ was to show that the $\beta_k(p)$ are continuous. We will not need continuity to construct our utility functions (the construction is entirely geometrical We need it to prove continuity of the utility functions and we need continuous differentiability to prove that the gradients of u^k exist and are monotonic and bounded away from zero on a compact set and twice continuous differentiability to prove concavity of the utility functions.

<u>Part 2</u>:

There exist functions $u^1, \ldots, u^\ell$ such that $\operatorname{Max}[u^k(x)$ such that $x \in \{x \in \tilde{X}^k \mid p^t x \leq 0\}]$ is uniquely attained at $\bar{x}^k(p)$ for all $p \in \mathring{R}_+^\ell$ and $k = 1, \ldots, \ell$.

As a first step, consider the special case where $\beta_k(p) = 1$ for all p. Then $\bar{x}^k(p) = \Pi_{T(p)} e^k$.

<u>Lemma 1</u>: Let $\bar{u}_y(x) = -\|x-y\|^2 = -\sum_{k=1}^{\ell} (x_k - y_k)^2$. Then $\tilde{u}_y$ is a monotonic strictly concave utility function on $X = \{x \in \mathbb{R}^\ell \mid x_i < y_i,\ i = 1, \ldots, \ell\}$ whose derived rational individual excess demand $z(p)$ is exactly $\Pi_{T(p)} y$ for all $p \in \mathbb{R}^\ell$ with $p^t y > 0$.

<u>Proof</u>: $\tilde{u}_y(x)$ is exactly the negative of the square of the distance between x and y. Maximizing $\bar{u}_y(x)$ on $\{x \in \mathbb{R}^\ell \mid p^t x \leq 0\} \equiv B(p)$ is equivalent to minimizing the distance between y and $B(p)$, which obviously occurs uniquely at $\Pi_{T(p)} y$ so long as $p^t y \geq 0$. See

Figure 2. Moreover on $X, \tilde{u}_y$ is differentiable and $D\tilde{u}_y(x) = D\left[-\sum_{i=1}^{\ell} (x_i - y_i)^2\right] = -2(x_1-y_1, \ldots, x_\ell - y_\ell) \gg 0$ since $x \ll y$, hence $\tilde{u}_y$ is monotonic. Furthermore, $D^2\tilde{u}_y(x) = -2I$, hence $\tilde{u}_y$ is also strictly concave.

Q.E.D.

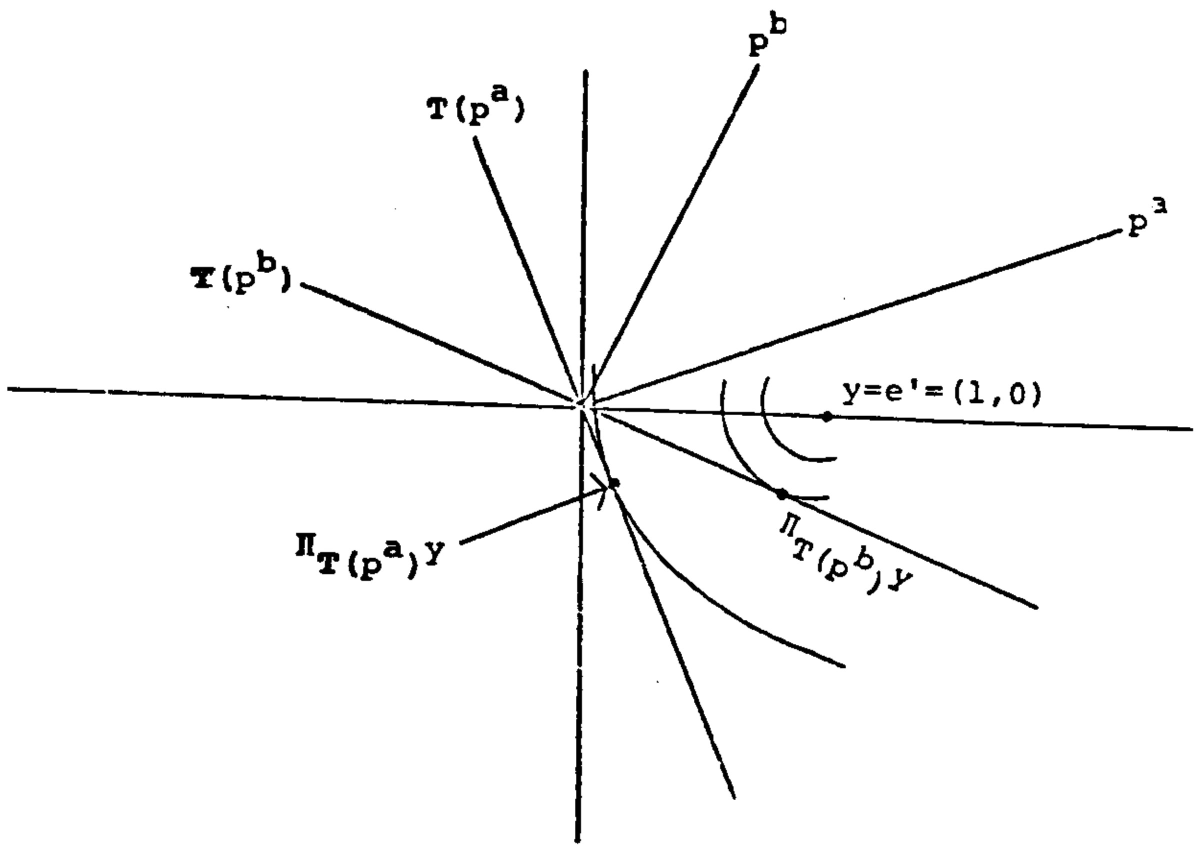
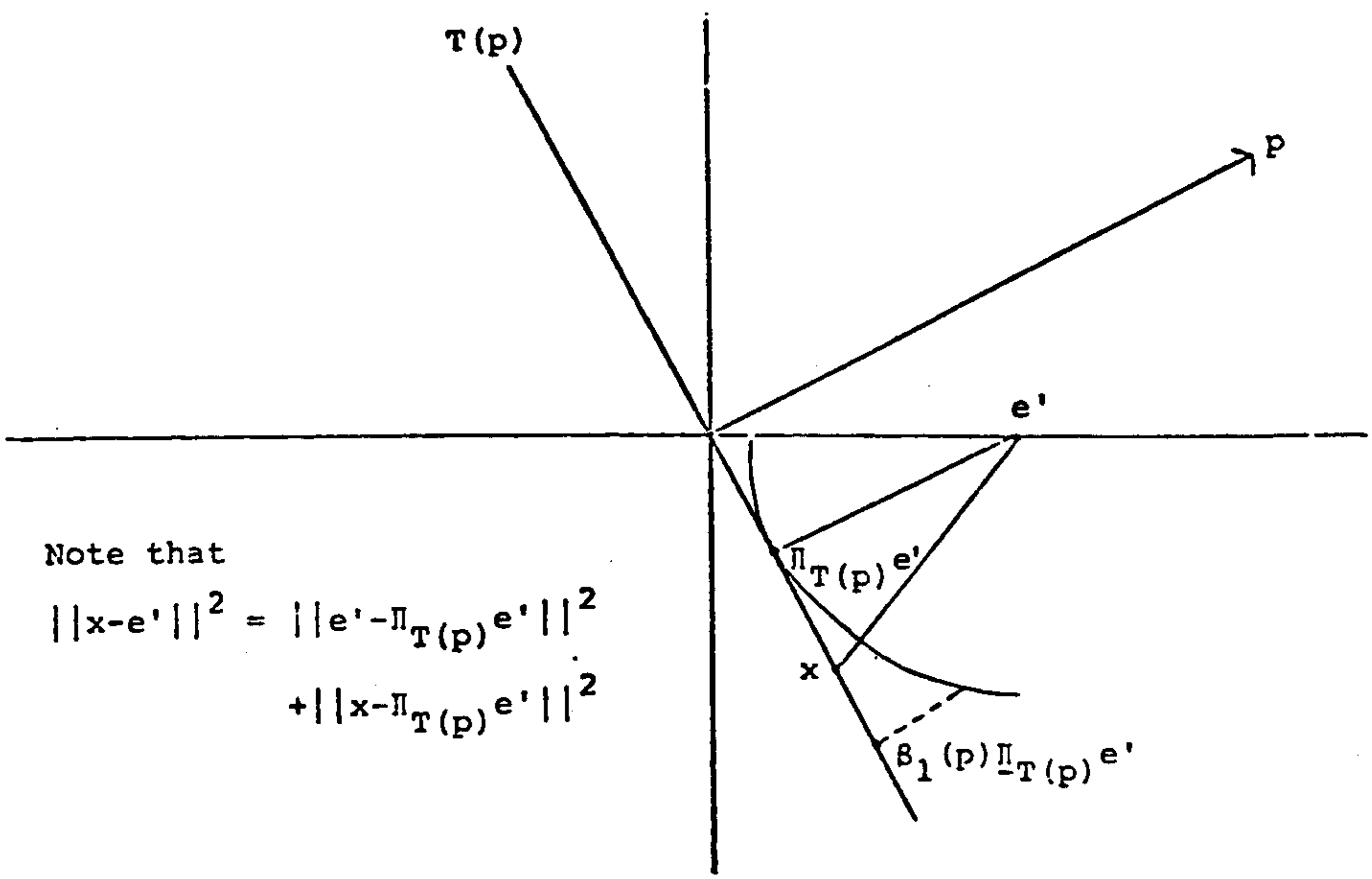


Figure 2

Since $p^t e^k > 0$ for all $p \in \mathring{R}^\ell_+$ and $k = 1, \ldots, \ell$, it follows that $z^k(p) \equiv \Pi_{T(p)} e^k$ is a rational individual excess demand function that can be derived from a utility function $\tilde{u}_k(x) = -||x-e^k||^2$, $k = 1, \ldots, \ell$.

All that remains is to prove that we can modify $\tilde{u}^k(x)$, getting $u^k(x$ such that the solution to Max $\{u^k(x) \mid x \in \tilde{X}^k$ and $p^t x \leqq 0\}$ is $\bar{x}^k(p) = \beta_k(p)\Pi_{T(p)}e^k$ rather than $\Pi_{T(p)}e^k$.

The Debreu construction of the indifference curves can be thought of as the bending of the circles centered at e^k until they are tangent to $T(p)$ at $\beta_k(p)\Pi_{T(p)}e^k$ instead of at $\Pi_{T(p)}e^k$. See Figure 3.



Note that

$$||x-e'||^2 = ||e'-\Pi_{T(p)}e'||^2 + ||x-\Pi_{T(p)}e'||^2$$

Figure 3

Debreu proved that this is possible on $\overset{+}{\mathbb{R}}^{\ell}_{\varepsilon} = \{p \in \overset{\circ}{\mathbb{R}}^{\ell}_{+} | \frac{p_i}{||p||} \geq \varepsilon\}$. We shall show it is actually possible on all of $\overset{\circ}{\mathbb{R}}^{\ell}$ by defining a utility function on all $\tilde{X}^k = \{x \in \mathbb{R}^{\ell} | x_i < 0,\ i \neq k,\ x_k > 0\}$. The idea is that given any $x \in \tilde{X}^k$ we can find a unique price vector $p(x) \in \overset{\circ}{\mathbb{R}}^{\ell}_{+}$ and a unique multiple $\lambda(x)x$ of x such that $\lambda(x)x = \Pi_{T(p(x))}e^k$. From Pythagoras' Law we have with this notation $\tilde{u}_k(x) = -||x - e^k||^2 = -(||\Pi_{T(p(x))}e^k - e^k||^2 + ||x - \Pi_{T(p(x))}e^k||^2) = -(||\lambda(x)x - e^k||^2 + ||x - \lambda(x)x||^2)$. We can get the correct excess demand by defining $u^k(x) = -(||\Pi_{T(p(x))}e^k - e^k||^2 + ||x - \beta_k(p(x))\Pi_{T(p(x))}e^k||^2)$.

<u>Proof of Part 2</u>: The ray $\{\lambda x | \lambda > 0\}$ and the point e^k not on the line (since $\lambda x_i < 0$ and $e^k_i = 0$ for $i \neq k$) determine a plane. Hence there is a unique line perpendicular to x through e^k, namely $p(x) = e^k - x_k \frac{x}{|x|^2}$. Observe that $p(x)^t x = x_k - x_k \frac{|x|^2}{|x|^2} = 0$ and that letting $\lambda(x) = \frac{x_k}{|x|^2}$, $\lambda(x)x + p(x) = e^k$ hence indeed $\Pi_{T(p(x))}e^k = \lambda(x)x$. See figure 4. It can be seen that the vector p is the vector of residuals derived from the regression of e^k onto X and that the first term in the above expression is simply the mean squared error of that regression, namely

$$- ||\frac{x^t e^k}{|x|^2}x - e^k||^2 = -||\frac{x_k x}{|x|^2} - e^k||^2 = -\left(\frac{x_k^2}{|x|^2} - 2\frac{x_k^2}{|x|^2} + 1\right) = \frac{x_k^2}{|x|^2} - 1.$$

Geometrically, as p varies through $\overset{\circ}{R}^{\ell}_{+}$, $\Pi_{T(p)}e^k$ traces out the hemisphere with center at $\frac{1}{2}e^k$. Given any $x \in \bar{X}^k$, the line from the origin through x intersects that hemisphere at exactly one point, $\lambda(x)x$. The line segment from $\lambda(x)x$ to e^k is $p(x)$. Recall from elementary geometry that two lines connecting the two endpoints of the diameter of a (semi)circle to the same point on the semicircle must meet at a right angle.

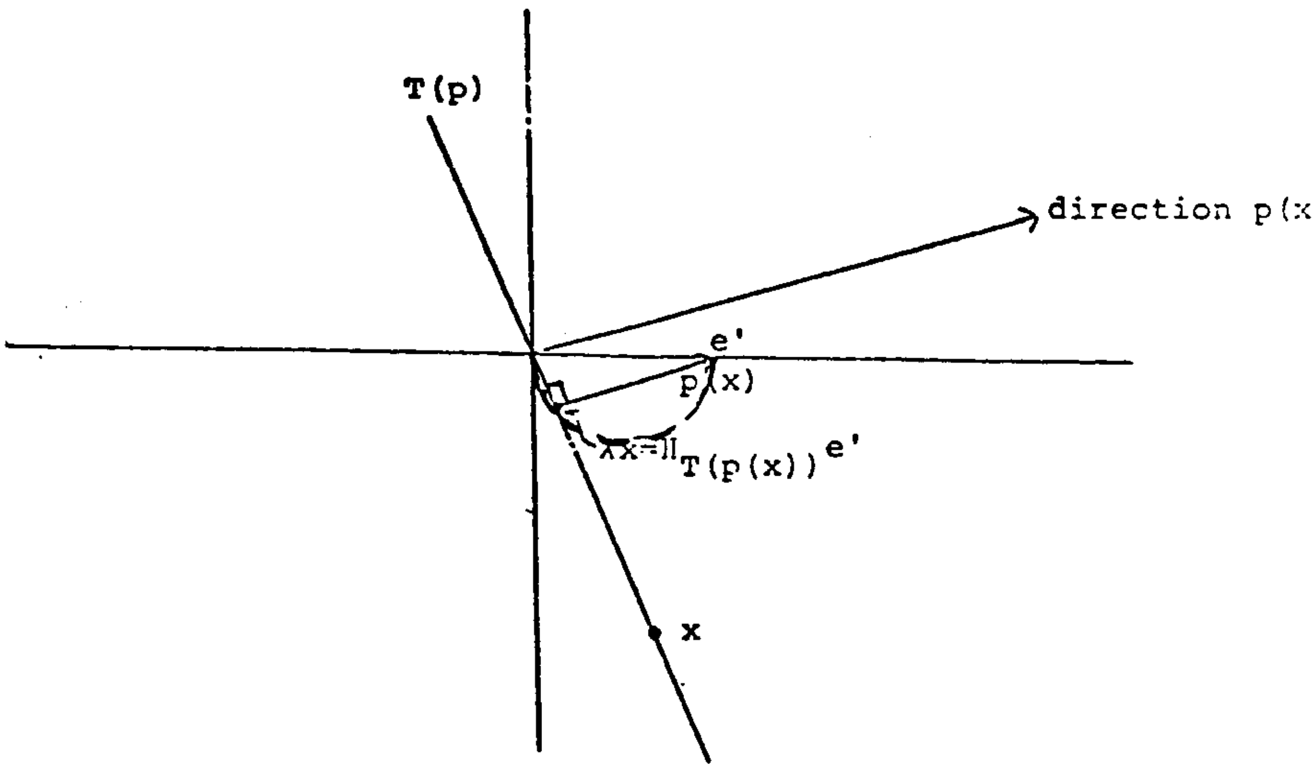


Figure 4

Observe that p is uniquely determined up to positive scalar multiples by the line x and the point e^k. Note that in fact p is a differentiable function of x. Note that $\tilde{X}^k$ was chosen so that the uniquely defined $p(x)$ is strictly positive for all $x \in \tilde{X}^k$.

Observe that if $x = \Pi_{T(\bar{p})} e^k$, then $p(x)$ is indeed $\bar{p}$, and if $x = \beta \Pi_{T(\bar{p})} e^k$ then also $p(x) = \bar{p}$.

Thus we can define

$$u^k(x) = -||\Pi_{T(p(x))}e^k - e^k||^2 - ||x - \beta_k(p(x))\Pi_{T(p(x))}e^k||^2$$

From the above demonstration we know that the problem Max $u^k(x)$ such that $x \in \tilde{X}^k$ and $p^t x \leq 0$ is solved uniquely by $\beta_k(p)\Pi_{T(p)}e^k = \bar{x}^k(p)$ since $p(\bar{x}^k(p)) = p$ and any multiple of $\Pi_{T(p)}e^k$ maximizes the first term of u^k subject to the budget constraint and $\bar{x}^k(p)$ uniquely maximizes the second term by making it zero.

Q.E.D.

<u>Part 3</u>: Now let

$$u^k(x) = -||\Pi_{T(p(x))}e^k - e^k||^2 - \varepsilon e^{n||x-\beta_k(p(x))\pi_{T(p(x))}e^k||^2}$$

For ε small enough and n big enough, this is monotonic and strictly quasi-concave on a compact, convex set X^k containing $\{\bar{x}^k(p)|p\varepsilon P\}$.

To prove this part notice first that the derived demands are unaffected. Now suppose P is compact in $\mathring{R}^\ell_+$; then $\{\bar{x}^k(p)|p\varepsilon P\}$ is compact if $\bar{x}^k$ is C°, hence we can find a closed convex, bounded X^k such that $\{\bar{x}^k(p)\ |p\varepsilon P\} \subset x^k \subset \tilde{x}^k$. Then if x is C^1, so that $\beta_k(p)$ is C^1, the gradient of $||x-\beta_k(p(x))\Pi_{T(p(x))}e^k||^2$ is bounded from below on X^k. But it is obvious that the gradient of $-||\Pi_{T(p(x))}e^k-e^k||^2$ is proportional to $p(x) >> 0$ and hence is bounded from below by a strictly <u>positive</u> constant in every coordinate on X^k. Hence for all sufficiently small $\varepsilon > 0$, $u^k(x)$ is monotonic on X^k and gives rise to the excess demand $\bar{x}^k(p)$.

The reader should also suspect from the geometry of figure 3 that the expression $v(x) = -||\Pi_{T(p(x))}e^k - e^k||^2 = \frac{x_k^2}{|x|^2} - 1$ has a second derivative D^2v which is negative definite on $[p(x),x]^{\perp}$ and identically 0 in the direction x. On the other hand, by taking n large we can make the second expression $w_n(x) = -e^{n||x - \beta(p(x))\lambda(x)x||^2}$ have a large negative definite second derivative $x'D^2w_n(x)x$ in the direction x so that the sum $D^2v+\varepsilon D^2w_n$ is negative definite on all $[Du^k]^{\perp} = [Dv + \varepsilon Dw_n]^{\perp}$ for ε sufficiently small, and therefore u^k is strictly quasi-concave. The point of taking n large is to make $|xD^2W_nx| >> |DW_n|$, as will become clear when we explicitly take derivatives to verify our intuition. That will conclude our proof of Part 3. The calculation of derivatives that follows is rather tedious; the reader may prefer to move directly to Part 4.

Consider any $\bar{x} \in X^k$, and the associated $\bar{p} = p(\bar{x})$.

To enable us to see clearly what is going on we shall change the basis, using $Q = (q^1, q^2, \ldots, q^{\ell})$ as our new orthonormal basis where we can assume $q^1 = \frac{\bar{p}}{|\bar{p}|}$ and $q^2 = \frac{\bar{x}}{|\bar{x}|}$. In the new basis $x_Q = Q^{-1}x = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell})^t$. Of course $\bar{x}_Q = Q^{-1}\bar{x} = (0, |\bar{x}|, 0, \ldots, 0)^t$ and $\bar{p}_Q = Q^{-1}\bar{p} = (|\bar{p}|, 0, \ldots, 0)^t$. Note also that $e^k_Q = Q^{-1}e^k = (q^1_1, q^2_1, 0, \ldots, 0)^t$ where $q^1_1 = e^k \cdot \frac{\bar{p}}{|\bar{p}|} = \frac{\bar{p}_k}{|\bar{p}|} > 0$ and $q^2_1 = e^k \cdot \frac{\bar{x}}{|\bar{x}|} = \frac{\bar{x}_k}{|\bar{x}|} > 0$ since by construction e^k lies in the same plane as p and x.

Consider first $v(x) = \frac{x_k^2}{|x|^2} - 1 = \frac{(x^te^k)^2}{|x|^2} - 1$

$$= \frac{[(Q^{-1}x)^t(Q^{-1}e^k)]^2}{(Q^{-1}x)^t(Q^{-1}x)} - 1 = \frac{(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2}{|\alpha|^2} - 1$$ where $\alpha = Q^{-1}x$. So we can think of the function $V_Q(\alpha) = V(Q\alpha)$ and $\frac{d^2V_Q}{d\alpha} = Q^{-1}D^2VQ$ so that if $\frac{d^2V_Q}{d\alpha^2}$ is negative definite in some direction $y_\alpha = Q^{-1}y$ then D^2V will be negative definite in the direction $y = QQ^{-1}y$.

Now, at any point α,

$$\frac{\partial V_Q}{\partial\alpha_1} = \frac{2(\alpha_1 q_1^1 + \alpha_2 q_1^2)q_1^1}{\Sigma\alpha_i^2} - \frac{(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2 2\alpha_1}{(\Sigma\alpha_i^2)^2}$$

$$\frac{\partial V_Q}{\partial\alpha_2} = \frac{2(\alpha_1 q_1^1 + \alpha_2 q_1^2)q_1^2}{\Sigma\alpha_i^2} - \frac{(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2 2\alpha_2}{(\Sigma\alpha_i^2)^2}$$

$$\frac{\partial V_Q}{\partial\alpha_j} = \frac{-(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2\ 2\alpha_j}{(\Sigma\alpha_i^2)^2} \quad \text{for } j = 3,\ldots,\ell.$$

Notice first that at the point $\alpha = \bar{\alpha} = Q^{-1}\bar{x} = (0, \bar{\alpha}_2, 0,\ldots, 0)^t$,

$$\frac{\partial V_Q}{\partial\alpha_1} = \frac{2\bar{\alpha}_2 q_1^2 q_1^1}{\bar{\alpha}_2^2} = \frac{2q_1^2 q_1^1}{\bar{\alpha}_2} = \frac{2\bar{x}_k\bar{p}_k}{|\bar{x}|^2|\bar{p}|} > 0$$

and

$$\frac{\partial V_Q}{\partial\alpha_2} = \frac{2\bar{\alpha}_2 q_1^2 q_1^2}{\bar{\alpha}_2^2} - \frac{\bar{\alpha}_2^2 q_1^2 q_1^2 2\bar{\alpha}_2}{\bar{\alpha}_2^4} = \frac{2\bar{\alpha}_2 q_1^2 q_1^2}{\bar{\alpha}_2^2} - \frac{2\bar{\alpha}_2 q_1^2 q_1^2}{\bar{\alpha}_2^2} = 0$$

and for $j = 3,\ldots,\ell$

$\frac{\partial V_Q}{\partial\alpha_j} = 0$ at $\bar{\alpha} = (0, \bar{\alpha}_2, 0,\ldots, 0)^t$. Hence indeed $DV(\bar{x})$ is a positive multiple of $p(\bar{x})$.

Proceeding now to the second derivative, for $j = 3, \ldots, \ell$

$$\frac{\partial^2 V_Q}{\partial \alpha_j \partial \alpha_j} = \frac{-(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2 2}{(\Sigma \alpha_i{}^2)^2} + \frac{(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2 4\alpha_j^2}{(\Sigma \alpha_i^2)^3}$$

which at the point $\bar{\alpha} = Q^{-1}\bar{x} = (0, \bar{\alpha}_2, 0, \ldots, 0)$

$$= \frac{-(\alpha_2 q_1^2)^2 2}{\alpha_2{}^4} = \frac{-(q_1^2)^2 2}{\alpha_2^2} < 0 \quad .$$

For $j, k = 3, \ldots, \ell$, $j \neq k$

$$\frac{\partial^2 V_Q}{\partial \alpha_j \partial \alpha_k} = \frac{(\alpha_1 q_1^1 + \alpha_2 q_1^2)^2 4\alpha_j \alpha_k}{(\Sigma \alpha_i)^3} = 0 \text{ at } \alpha = (0, \bar{\alpha}_2, 0, \ldots, 0)$$

and similarly for $j = 3, \ldots, \ell$ and $k = 1$ or 2,

$$\frac{\partial^2 V_Q}{\partial \alpha_j \partial \alpha_k} = \frac{\partial^2 V_Q}{\partial \alpha_k \partial \alpha_j} = 0 \text{ at the point } \alpha = (0, \bar{\alpha}_2, 0, \ldots, 0).$$

Finally, at the point $\bar{\alpha} = (0, \bar{\alpha}_2, 0, \ldots, 0)$

$$\frac{\partial^2 V_Q}{\partial \alpha_2 \partial \alpha_2} = \frac{2q_1^2 q_1^2}{\alpha_2{}^2} - \frac{4\alpha_2^2 q_1^2 q_1^2}{\alpha_2{}^4} - \frac{4\alpha_2{}^2 q_1{}^2 q_1^2}{\alpha_2{}^4} - \frac{2\alpha_2^2 q_1^2 q_1^2}{\alpha_2{}^4} + \frac{8\alpha_2{}^2 q_1^2 q_1^2 \alpha_2{}^2}{\alpha_2{}^6} = 0$$

$$\frac{\partial^2 V_Q}{\partial \alpha_1 \partial \alpha_2} = \frac{2q_1^2 q_1^1}{\alpha_2{}^2} - \frac{4\alpha_2{}^2 q_1^2 q_1^1}{\alpha_2{}^4} = -\frac{2q_1^2 q_1^1}{\alpha_2{}^2} < 0$$

and $$\frac{\partial^2 V_Q}{\partial \alpha_1 \partial \alpha_1} = \frac{2q_1^1 q_1^1}{\alpha_2{}^2} - \frac{2\alpha_2{}^2 q_1^2 q_1^2}{\alpha_2{}^4} = \frac{2}{\alpha^4} (q_1^1 q_1^1 - q_1^2 q_1^2)$$

which is positive for some $\bar{x}$ and negative for others.

Thus at $\alpha = \bar{\alpha} = Q^{-1}\bar{x} = (0, \bar{\alpha}_2, 0, \ldots, 0)$,

$$D^2V_Q = \begin{bmatrix} a & b & & \\ b & 0 & & \bigcirc \\ & & -c & \\ \bigcirc & & & -c \end{bmatrix}$$

where $b < 0$, $c > 0$.

Now consider $W_n(x) = -e^{n(f(x))^2}$ where $f(x) = ||x - \beta(p(x))\Pi_{T(p(x))}e^k||$.

$$DW_n(\bar{x}) = -2nf(\bar{x})e^{n(f(\bar{x}))^2}\nabla f(\bar{x})$$

so $|DW_n(\bar{x})| = 2n|f(\bar{x})||e^{n(f(\bar{x}))^2}||\nabla f(\bar{x})|$

Now from the special structure of $f(x)$ it is easy to see that in the new coordinates $\frac{\partial W^2_{nQ}}{\partial\alpha_2\partial\alpha_2} = -2ne^{n(f(\bar{x}))^2}$

$-4n^2(f(\bar{x}))^2e^{n(f(\bar{x}))^2} < 0$

Note that $\left|\frac{\partial W^2_{nQ}}{\partial\alpha_2 2\alpha_2}\right| > 2n$ and

$$\frac{\left|\frac{\partial W^2_{nQ}}{\partial\alpha_2\partial\alpha_2}\right|}{|DW_n(\bar{x})|} = \frac{1}{|f(\bar{x})||\nabla f(\bar{x})|} + 2n|f(\bar{x})| \;\; / \;\; |\nabla f(\bar{x})|$$

and on compact sets this can be made arbitrarily large by taking n big enough.

$$D^2u^K_Q = D^2V_Q + \epsilon D^2W_{nQ} = \begin{bmatrix} a & b & & \\ b & 0 & & \bigcirc \\ & & -c & \\ \bigcirc & & & -c \end{bmatrix} + \epsilon \begin{bmatrix} w_{11} & w_{12} & \cdots \\ w_{21} & -B & \cdots \\ \vdots & \vdots & \end{bmatrix}$$ where $B = \frac{\partial^2 W^2_{nQ}}{\partial\alpha_2\partial\alpha_2} >$

Obviously, $\lim_{\varepsilon \to 0} Du^k = \bar{p}$ and since D^2u^k is clearly negative definite on the space $[q^3, \ldots, q^\ell]$ for ε small, we need only worry about vectors y, $|y| = |DV|$, perpendicular to Du^k of the form $y = \gamma_1 \frac{\bar{p}}{|\bar{p}|} + \gamma_2 \frac{\bar{x}}{|\bar{x}|}$. Then $y^t D^2 u_Q^k y = a\gamma_1^2 + 2b\gamma_1\gamma_2 + \varepsilon W_{11}\gamma_1^2 + 2\varepsilon W_{12}\gamma_1\gamma_2 - \varepsilon B\gamma_2^2$. It is clear from diagram 5 that $|\gamma_1| \leq \varepsilon |DW_n|$ and $\gamma_1^2 + \gamma_2^2 = |DV|^2$

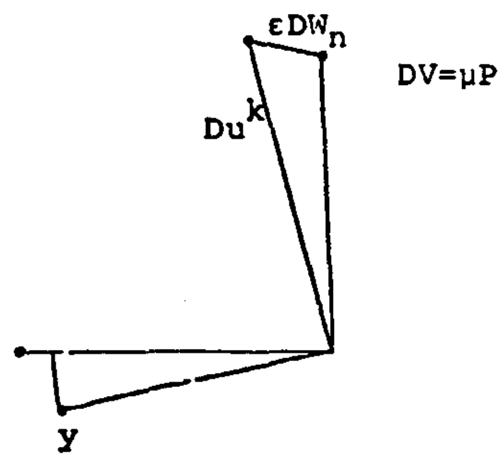


Diagram 5

hence $y^t D^2 u_Q^k \, y \leq 0(\varepsilon^2) + 2b\gamma_2 |DW_n| \varepsilon - \varepsilon B\gamma_2^2$.

Since we can take $B / |DW_n|$ arbitrarily large, we can make this negative and do it simultaneously for all the points $\bar{x} \in X^k$ since X^k itself is contained in a compact set, by choosing a large n and then choosing a very small ε.

Q.E.D.

Part 4: We can modify the utility functions u^k one more time so that they are strictly concave and monotonic on X^k, $k=1,\ldots,\ell$, and give rise to the same excess demands $\bar{x}^k(p)$ for $p\varepsilon P$. We can then extend the u^k to all of R^ℓ, preserving monotonicity and concavity. The $\bar{x}^k(p)$ will again maximize u^k subject to the budget constaint $p^t x \leq 0$, for all $p\varepsilon P$, $k=1,\ldots,\ell$.

Proof of Part 4: Fortunately the proof of Part 4 is quite easy. Aumann [1] demonstrated that it is always possible to monotonically transform a strictly quasi-concave C^2 function into a strictly concave function on any compact set X^k. Simply define $V^k(x) = -e^{-Nu^k(x)}$ for N big enough. Then $DV^k(x) = NDu^k e^{-Nu^k}$ and $D^2V^k = [ND^2u^k - N^2Du^k(Du^k)^t]e^{-Nu^k}$ which is negative definite on $[Du^k]^{\perp}$ since D^2u^k is, and for N big enough is clearly negative definite on all of R^ℓ.

To extend V^k to all of R , define $\bar{u}^k$ as the infimum of all linear functions that lie above V^k on all of X^k. Formally, for every $y\varepsilon X^k$, let Ly be the linear function $Ly(x) = V^k(y) + DV^k(y)^t(x-y)$. Since V^k is C^2 and concave on X^k, $Ly(x) \geq V^k(x)$ for all $x\varepsilon X^k$ and $Ly(y) = V^k(y)$. Let $\bar{u}^k(x) = \inf\{Ly(x)\,|\,y\varepsilon X^k\}$. Since the inf of concave (linear) functions is concave, $\bar{u}^k$ is concave. Moreover, $\bar{u}^k(x) = V^k(x)$ for all $x\varepsilon X^k$. Furthermore, since X^k is compact and V^k continuously differentiable, $\bar{u}^k$ is well-defined and finite on all R^ℓ.

Q.E.D.

Finally, observe that it is possible to choose $w^k \varepsilon R_+^\ell$ such that $w^k >> \bar{x}^k(p)$ for all $p\varepsilon P$. In that case we can more traditionally restrict the feasible net trade space to $R_+^\ell - w^k$ without disturbing the maximization of utility for $p\varepsilon P$. If we had begun with an observable aggregate endowment w as well as $x(p)$, this last argument would not in general be valid. There would indeed be restrictions on community excess demands (at least $x(p) + w \geq 0$ for all p).

Bibliography

Aumann, R: "Values of Market Games with a Continuum of Traders." *Econometrica*, 1975.

Debreu, G: "Excess Demand Functions." *Journal of Mathematical Economics* 1, 1974.

McFadden, Mas-Collel, Mantel, Richter: "A Characterization of Community Excess Demand Functions." *Journal of Economic Theory* 9, 1974.

Mantel, R. R.: "On the Characterization of Aggregate Excess Demands." *Journal of Economic Theory* 7, 1974.

Mantel, R. R.: "Homothetic Preferences and Community Excess Demand Functions." *Journal of Economic Theory* 12, 1976.

Sonnenschein, H.: "Do Walras Identity and Continuity Characterize the Class of Community Excess Demand Functions?" *Journal of Economic Theory* 6, 1973.